LA-UR -79-3326

CONF - 791157--1

TITLE: SOME REMARKS ON THE RESOLUTION OF THE MULTIPLICITY

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SUBMITTED TO: Conference on Group Theoretical Methods in Physics, November 28-30, 1979
Zvenigorod, U.S.S.R.

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UNITED STATES
DEPARTMENT OF ENERGY
CONTRACT W-7405-ENG. 36

19 November 1979

Some Remarks on the Resolution of the Multiplicity Problem for Tensor Operators in U(n)

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Abstract

A canonical resolution of the multiplicity problem has been proven for U(3) and the present paper extends this resolution to a determination of all U(n) tensor operators characterized by maximal null space.

Invited paper for the Conference on Group Theoretical Methods in Physics, 28-30 November 1979, Zvenigorod, U.S.S.R.

*Research supported in part by the National Science Foundation and the Army Research Office.
**Research supported by the United States Department of Energy

I. Introduction:

Group theoretical techniques are of the greatest importance in the applications of unitary symmetry in quantum physics, and--in one form or another--the typical problem involves the construction of the unit tensor operators (Wigner operators) for the relevant group. For the groups {U(n)} the construction of unirreps (unitary irreducible representations) is known from the work of Gel'fand among others;[2][3] the construction of unit tensor operators is equivalent to the explicit construction of the unitary matrix which brings the general Kronecker product to diagonal form. As is well-known the occurence of multiplicity introduces ambiguity into this reduction. The resolution of this multiplicity problem may be looked upon as the problem of finding a suitable analog for tensor operators in U(n), to the Weyl branching law for unirreps in U(n), since it is this branching law which validates the canonical[#1] resolution of the labelling problem for vectors of an irrep by means of the Gel'fand pattern.

For U(3) such a canonical resolution is known.[4] Unit tensor operators are canonically labelled by three arrays: a) the irrep carried by the operator is labelled by $[M_{13},M_{23},M_{33}]$ (which specifies the Young frame) b) the specific vector in the irrep is labelled by a Gel'fand pattern $\begin{pmatrix} M_{12} & & M_{22} \\ & M_{11} & \end{pmatrix}$ and c) the specific operator in the multiplicity set is labelled by an operator pattern, $\begin{pmatrix} & \Gamma_{11} & \\ \Gamma_{12} & & \Gamma_{12} \end{pmatrix}$ whose weights specify the shifts $\Delta_i(\Gamma)=M_{i3}^{\text{final}}-M_{i3}^{\text{initial}}$ induced by the operator when acting on an irrep (m) in U(3).

The significance of the operator patterns (Γ) in effecting the resolution can best be seen from the normal form for unit tensor operators (more particularly, for the projective matrix elements, (see equation (11) below.) One sees that this normal form associates a unique operator to each lexical operator pattern and effects a mapping: $\left\langle\begin{smallmatrix}(r)\\ [M]\end{smallmatrix}\right\rangle \to D^2\left(\begin{smallmatrix}(r)\\ [M]\end{smallmatrix}\right)$ of the canonical operator $\left\langle\begin{smallmatrix}(r)\\ [M]\end{smallmatrix}\right\rangle$ to a function (invariant under the group action) which vanishes on all irreps of the characteristic null space. #2

It is conjectured that these features of the known U 3) resolution generalize to all U(n). The purpose of the present paper is to show that this conjecture is valid for all U(n) for all operators characterized by maximal null space. An explicit canonical construction of the denominator function $D\left(\begin{smallmatrix}(r_s)\\ [M]\end{smallmatrix}\right)$ which characterizes all such operators is given in Section III (see equation (18)). Knowledge of this denominator function suffices to determine the operators themselves (confer Section III, remark (7)).

II. Technical Preliminaries

A. The Imbedding $U(n^2) \supset U(n) \times U(n) \supset U(n)$

We will use the techniques of boson operators (Heisenberg group). The basic underlying idea is the *Jordan map:* given a set of nxn matrices $\{\Lambda\}$ over $\mathbb{C}$, define the mapping:

$$\mathcal{J}: \Lambda = (\Lambda_{ij}) \to \Lambda_{op} = \sum_{i,j} (\Lambda_{ij}\, a_i \bar{a}_j), \tag{1}$$

where the $\{a_i\},\{\bar{a}_i\}$ for $i=1,\ldots,n$ are boson operators obeying the defining relations:

$$[a_i, a_j] = [\bar{a}_i, \bar{a}_j] = 0; \qquad [\bar{a}_i, a_j] = \delta_{ij}, \; i=1,2,\ldots,n \tag{2}$$

The mapping $\mathcal{J}$ has the property that it preserves commutation relations:

$$[\Lambda^{(1)}_{op}, \Lambda^{(2)}_{op}] = [\Lambda^{(1)}, \Lambda^{(2)}]_{op}. \tag{3}$$

We can extend this concept by considering in place of the n component boson the nxn boson operator a^i_j, or *matrix boson operator* $A = (a^i_j)$. These n^2 operators obey:

$$[a^i_j, a^{i'}_{j'}] = [\bar{a}^i_j, \bar{a}^{i'}_{j'}] = 0, \; [\bar{a}^i_j, a^{i'}_{j'}] = \delta^{i'}_i \delta^{j'}_j \quad . \tag{4}$$

Two different Jordan maps are now defined:

$$J_u \equiv J_{lower}: \; (\Lambda) = (\Lambda_{ij}) \to J_\ell(\Lambda) = \sum_{k=1}^{n} \Lambda_{ij} a^k_i \bar{a}^k_j \quad , \tag{5a}$$

$$J_\ell \equiv J_{upper}: \; (\Lambda) = (\Lambda_{ij}) \to J_u(\Lambda) = \sum_{k=1}^{n} \Lambda_{ij} a^i_k \bar{a}^j_k \quad . \tag{5b}$$

Both maps preserve commutation relations. Note that *all* $\mathcal{J}_u(\Lambda)$ *commute with all* $\mathcal{J}_\ell(\Lambda)$. Hence if the $\{\Lambda\}$ are the generators of a fundamental irrep of U(n), the extended Jordan map yields generators of the group U(n)xU(n). Letting (Λ) include the $n^2 x n^2$ matrices of a fundamental irrep of $U(n^2)$ we obtain generators for all totally symmetric irreps of $U(n^2)$ adapted to the group decomposition: $U(n^2) \supset U(n) \times U(n)$.

Let us now determine more explicitly the irreps defined by this procedure. Using the elements of the matrix boson A as indeterminates, we consider homogeneous polynomials as the basis. These are the *boson polynomials*,[3] denoted by:

$$B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(A) \quad ,$$

where A is the matrix boson (a^i_j), and $[m]$ denotes the Young pattern labels $[m_{1n},m_{2n},\ldots,m_{nn}]$ with (m) and (m') being Gel'fand patterns of n-1 rows. The operators $\mathcal{J}(A)$ act on the boson polynomials by commutation; by construction the boson polynomials are the vectors denoted by the respective Gel'fand patterns of the group.

Example: If the Gel'fand patterns (m) and (m') are maximal then

$$B\begin{pmatrix}(\max)\\ [M]\\ (\max)\end{pmatrix}(A) = \prod_{k=1}^{n} \left(a^{12\ldots k}_{12\ldots k}\right)^{M_{kn}-M_{k+1n}} \quad , \tag{6}$$

where $a^{1\ldots k}_{1\ldots k}$ is the determinant formed from the k bosons a^i_j, $i,j\leq k$.

Remarks:

(1) By construction the boson polynomials obey the product law:

$$B\begin{pmatrix}(\mu')\\ [m]\\ (\mu)\end{pmatrix}(\tilde{X}AY) = \sum_{(m)(m')} B\begin{pmatrix}(\mu)\\ [m]\\ (m)\end{pmatrix}(X)\, B\begin{pmatrix}(m')\\ [m]\\ (m)\end{pmatrix}(A)\, \begin{pmatrix}(\mu')\\ [m]\\ (m')\end{pmatrix}(Y) \tag{7}$$

(Here the tilde denotes matrix transposition.)

(2) The boson variables A appearing in B(A) are indeterminates. If we specialize $A\to U$ with U an nxn unitary matrix we see (from the product law) that the $B([m])(U)$ are finite dimensional unitary matrix representations of U(n). (That they define irreps labelled by [m] follows from the action of the operators $\mathcal{J}(A)$.) We are, however, free to interpret the indeterminates A as elements of an arbitrary non-singular complex matrix. Just as in remark (1) we can conclude that *these same polynomials are matrix irreps of* GL(n,C).

B. <u>The factorization lemma</u>:

We have seen that this matrix boson realization involves the direct product group U(n)xU(n). One sees in fact that this boson realization really involves the group $U(n^2)$ and all totally symmetric irreps thereof. This defines an imbedding of U(n) in the sequence of groups $U(n^2) \supset U(n) \times U(n) \supset U(n)$, in which, moreover, the irrep labels of the two U(n) groups in U(n)xU(n) coincide [we denote this by U(n)*U(n)]. This structure is the analog to that exhibited by the tensor operators of U(n), and we exploit this formal analogy to discuss the factorization lemma.[4]

Let $\left|\begin{pmatrix}(M')\\ [M]\\ (M)\end{pmatrix}\right\rangle$

denote a normalized basis vector in an irrep space of U(n)*U(n). In this notation, the first U(n) refers to the U(n) group with generators $\mathcal{J}_u(\Lambda)$, the second to the U(n) group with generators $\mathcal{J}_\ell(\Lambda)$. These two U(n) groups are isomorphic but distinct (and commuting); the placement of the indices is merely a reminder as to which group is which ("upper" vs. "lower") - there is no other implication.

The star signifies that the Casimir invariants of the irreps of these two groups conicide. Hence, both

$$(M) = \begin{pmatrix}[M]\\ (M)\end{pmatrix} \quad \text{and} \quad (M') = \begin{pmatrix}(M')\\ [M]\end{pmatrix}$$

are Gel'fand patterns, the second one being inverted. The basis vectors may also be written in the form

$$\left|\begin{pmatrix}(M')\\ [M]\\ (M)\end{pmatrix}\right\rangle = \mathcal{M}([M])^{-\frac{1}{2}}\, B\begin{pmatrix}(M')\\ [M]\\ (M)\end{pmatrix}(A)|0\rangle \quad , \text{ where } \quad B\begin{pmatrix}(M')\\ [M]\\ (M)\end{pmatrix}(A)$$

is an operator-valued polynomial on the set of boson operators $A=\{a^i_j\}$, the symbol $|0\rangle$ denotes the vacuum ket, and $\mathcal{M}([M])$ is a function defined on the highest weight tableau associated with the irrep [M]:

$$\mathcal{M}([M]) \equiv \left(\prod_{i=1}^{n} (M_{in}+n-i)! \right) \times \left(\prod_{i<j=1}^{n} (M_{in}-M_{jn}+j-i) \right)^{-1} \qquad (8)$$

The introduction of $\mathcal{M}^{-\frac{1}{2}}$ defines the manner in which the basis vectors are normalized.

The boson polynomials are clearly tensor operators with respect to transformations in the respective U(n) subgroups of U(n)*U(n), denoted by the lower or upper Gel'fand pattern. As such, this operator relation must be bilinear in the canonical Wigner operators which are defined, respectively, on the two U(n) groups. *The factorization lemma asserts that the precise form of this bilinear relation is:*

$$B\begin{pmatrix} (M') \\ [M] \\ (M) \end{pmatrix} (A) = \mathcal{M}^{\frac{1}{2}} \sum_{(\Gamma)} \left\langle \begin{matrix} (\Gamma) \\ [M] \\ (M) \end{matrix} \right\rangle_{\ell} \left\langle \begin{matrix} (\Gamma) \\ [M] \\ (M') \end{matrix} \right\rangle_{u} \mathcal{M}^{-\frac{1}{2}} \,, \qquad (9)$$

where $\mathcal{M}$ is an invariant operator of U(n)*U(n) which has eigen-value (cf.eq.(8)) equal to $\mathcal{M}([M])$ for an arbitrary vector with labels [M]. The indices ℓ and u designate the fact that the Wigner operators act, respectively, on the lower and upper Gel'fand patterns of an arbitrary vector of U(n)*U(n). (Note that the two Wigner operators commute since they act in different spaces.)

Remarks:

(1) There is an important special case of equation (9):

$$a_i^j = \mathcal{M}^{\frac{1}{2}} \left(\sum_{\tau=1}^{n} \left\langle [1 \overset{\tau}{} \dot{0}] \atop i \right\rangle_{\ell} \left\langle [1 \overset{\tau}{} \dot{0}] \atop j \right\rangle_{u} \right) \mathcal{M}^{-\frac{1}{2}} \qquad (10)$$

where the $[1\dot{0}]$ denote the (unique) fundamental Wigner operators. It is this special case that accounts for the term "boson factorization".

(2) The result in equation (9) is important not only in affording a method to calculate Wigner operators but also in showing that, at this stage, all resolutions of the multiplicity are equivalent (since an arbitrary orthogonal

transformation leaves the (Γ) sum invariant.)

(3) When we use the factorization lemma below we will take the labels (Γ) to be the canonical labelling defined by null space.

C. Projective Operators

To simplify the discussion of the unit tensor operators in U(n) it is useful to employ recursive methods and consider that all U(n-1) operators are known.

We can them consider *projective operators,* $\left|\begin{matrix}(\Gamma)\\ [M]\\ (\Gamma')\end{matrix}\right|$, which are scalar products in U(n-1) of unit tensor operators $\left\langle\begin{matrix}(\Gamma)\\ [M]\end{matrix}\right\rangle$ in U(n) and $\left\langle\begin{matrix}(\Gamma')\\ [M]_{n-1}\end{matrix}\right\rangle$ in U(n-1).

Note that both patterns(Γ) and (Γ') in a projective operator are operator patterns.

It is a remarkable fact that the explicit matrix elements of all extremal #3 unit U(n):U(n-1) projective operators can be calculated from a few simple rules of the pattern calculus.[5] In particular, this class of explicitly known projective operators includes all elementary operators of the form $[\dot{1}_k \dot{0}_{n-k}]$ (a dot over a numeral implies that the numeral is repeated a number of times equal to the subscript), which themselves are a *product basis for constructing all* U(n) *tensor operators.*

We can now give a standard form for projective operators:

$$\begin{bmatrix}(\Gamma)\\ [M]\\ (\Gamma')\end{bmatrix} = \frac{\text{NPCF}}{\text{Denom}\begin{pmatrix}(\Gamma)\\ [M]\end{pmatrix} \cdot \ \text{Denom}_{n-1}\ (\Gamma')} \cdot \text{Polynomial}, \qquad (11)$$

where:

(1) The polynomial is over the variables $p_{i,n-1} \equiv m_{i,n-1} + n-1-i$ of the U(n-1) subgroup with the variables $p_{i,n}$ of the U(n) group as parameters. (The leading term is normed to 1.)

(2) The NPCF (Numerator pattern calculus factor) is an explicitly known square root of a product of linear factors in $\{p_{i,n}\}$ and $\{p_{i,n-1}\}$.

(3) Denom $_{n-1}(\Gamma')$ is a (recursively) known function of the $\{p_{i,n-1}\}$ and defined on the U(n-1) subgroup.

(4) The denominator function Denom $\begin{pmatrix} (\Gamma) \\ [M] \end{pmatrix}$ is a function of the $\{p_{i,n}\}$ alone, which is invariant under U(n) and totally symmetric under the S(n) symmetry group permuting the $\{p_{i,n}\}$ variables.

III. Determination of the Denominator Function for Operators having Maximal Null Space

We begin by recalling the form of the factorization lemma, eq. (9), which asserted that the general boson polynomial could be viewed as a tensor operator in U(n) x U(n):

$$B\begin{pmatrix}(M')\\ [M]\\ (M)\end{pmatrix}(A) = M^{\frac{1}{2}} \sum_{(\Gamma)} \left\langle\begin{matrix}(\Gamma)\\ [M]\\ (M)\end{matrix}\right\rangle_{\ell} \left\langle\begin{matrix}(\Gamma)\\ [M]\\ (M')\end{matrix}\right\rangle_{u} M^{-\frac{1}{2}} \ . \qquad (12)$$

Next we use the general form for the maximal boson polynomials expressed as a Cartan product of elementary operators:

$$B\begin{pmatrix}(\max)\\ [M]\\ (\max)\end{pmatrix}(A) = \prod_{k=1}^{n} (a^{12\ldots k}_{12\ldots k})^{M_{kn}-M_{k+1n}} \qquad (13)$$

Finally we use the fact that the matrix elements of all elementary operators are all explicitly known from the pattern calculus algorithm.[5] We use this information in the form:

$$a^{12\ldots k}_{12\ldots k} = M^{\frac{1}{2}} \Big(\sum_{\gamma_k} \langle [\dot{1}_k \overset{\gamma_k}{\underset{\max}{}} \dot{0}_{n-k}]\rangle_u \langle [\dot{1}_k \overset{\gamma_k}{\underset{\max}{}} \dot{0}_{n-k}]\rangle_{\ell}\Big) M^{-\frac{1}{2}} \ . \qquad (14)$$

We introduce this form for each of the elementary operators appearing in equation (13) above, and then equate the RHS of equation (12) to this result. Before writing out the resultant equation, we simplify it by making a few observations:

(a) The operator M cancels out on both sides and may be eliminated

(b) The boson operators in equation (13) all commute, hence we can take all possible orders and divide by the multinomial factor:

$$F(\mathrm{m.f.}) = M_{1n}! / \prod_{i=1}^{n} (M_{in}-M_{i+1n})! \qquad (15)$$

We find:

$$\sum_{\Gamma} \langle [M] \overset{\Gamma}{\underset{\max}{}} \rangle_u \langle [M] \overset{\Gamma}{\underset{\max}{}} \rangle_{\ell} = \frac{1}{F(\mathrm{m.f.})} \sum \prod \Big(\langle [\dot{1}_k \overset{\gamma_k}{\underset{\max}{}} \dot{0}_{n-k}]\rangle_u \langle [\dot{1}_k \overset{\gamma_k}{\underset{\max}{}} \dot{0}_{n-k}]\rangle_{\ell} \qquad (16)$$

where $\prod$ denotes a product of elementary operators, with the sum over all possible orders with all possible shifts.

To simplify the problem we convert both sides by equation (16) into <u>projection</u> operators by taking the scalar product with the unique maximal U(n-1) operator of maximal shift. Since for these operators the patterns simply add, we may use the appropriate product decomposition on the RHS.

We thus obtain:

$$\sum_{\Gamma} \begin{bmatrix} \Gamma \\ [M] \\ \max \end{bmatrix}_u \begin{bmatrix} \Gamma \\ [M] \\ \max \end{bmatrix}_\ell = \frac{1}{F(\mathrm{n.f.})} \sum \prod \left(\begin{bmatrix} & \gamma_k & \\ \dot{1}_k & & \dot{0}_{n-k} \\ & \max & \end{bmatrix}_u \begin{bmatrix} & \gamma_k & \\ \dot{1}_k & & \dot{0}_{n-k} \\ & \max & \end{bmatrix}_\ell \right) \quad (17)$$

We now come to the essential idea: to determine the null space of a unit tensor operator we need only the denominator function for that operator. Moreover this denominator function is dependent solely on the $U(n)$ invariant labels $[M]$.

Next we recognize that Equation (17) taken between fixed initial and final states is an algebraic identity valid for arbitrary values of the indeterminates.

Accordingly, we will consider Equation (17) to be taken between fixed initial and final irrep labels and apply the pattern calculus rules.

We sketch the derivation now by a series of comments:

(1) Since the $U(n-1)$ operator has maximal shift there is but one way to write the required shift in each elementary operator ($U(n-1)$ part). By contrast, the shift in $U(n)$--since a general shift--has many ways of being written.

(2) We now let $p_{n-1,n-1}$ become large and positive for each elementary operator on the RHS. (We take $p_{n-1,n-1}$ in both upper and lower irreps to be large.) Now observe that for *each* elementary operator the limit is independent of the shift γ_k. (Recall that the $U(n-1)$ shift is fixed.)

(3) We repeat the limiting operation for each of the $p_{i,n-1}$ variables in turn, keeping the overall powers of these variables.

(4) Since we perform the limits in both the upper and lower patterns all phases are positive.

(5) The net result is that all numerator pattern calculus factors and $U(n-1)$ pattern calculus denominators become independent of the shifts in $U(n)$, and independent of where they appear in the product.

(6) Clearly the RHS approaches a single result, i.e., the space of operator patterns in $U(n)$ becomes 1-dimensional.

(7) From (6) it follows that the LHS is restricted to a single term. We identify this term as the pattern Γ_s of greatest null space. (This identification requires proof, see remark (2) below.)

(8) All limiting factors cancel on both sides leaving only the denominator functions in U(n).

The net result of these observations is to validate the following form for the denominator function for the general unit tensor operator $\langle [M] \rangle$ in U(n) having maximal null space (denoted by the "stretched" operator pattern Γ_s), cf. remark (4) below.)

Path sum formula:

$$\left[C\begin{pmatrix} (\Gamma_s) \\ [M] \end{pmatrix} ([m]_n) \right]^{-2} = \sum_{\substack{\text{paths} \\ \{\gamma_i\}}} \left[\left(\prod_{\substack{\text{elementary} \\ \text{operators}}} \left[D\begin{pmatrix} \gamma_i \\ \dot{1}_k \, \dot{0}_{n-k} \end{pmatrix} ([m]_n) \right]^{-2} \right) \right] \tag{18}$$

where:

(a) the product denotes a product of squares of elementary denominator operators in any order subject to the constraint that precisely $M_{k,n}-M_{k+1,n}$ operators be of type $[\dot{1}_k \dot{0}_{n-k}]$ for every k (k = 1,2, . . . n), where the M_{in} are the Young frame labels of the tensor operator [M] (and $M_{n+1,n} \equiv 0$);

(b) the product $\prod$ acts on the irrep labels $[m]_n$ of the state vector in U(n);

(c) the $\sum$ over all paths denotes a sum over all sequences of operator pattern labels $\{\gamma_i\}$ of the denominator operators subject to the constraint that the shifts $\Delta(\gamma_i)$ induced by the operators obey the rule:

$$\sum_i \Delta(\gamma_i) = \Delta(\Gamma_s), \tag{19}$$

that is, the total shift induced by the product agrees with the shift associated with the tensor operator

$$\left\langle \begin{matrix} (\Gamma_s) \\ [M] \end{matrix} \right\rangle$$

Remarks:

(1) For simplicity a purely numerical normalizing factor has been omitted in the path sum formula.

(2) To validate that this result corresponds to maximal null space one observes that the path sum formula has precisely those linear factors in the denominator (poles) corresponding to the set of lines in the intertwining number-null space diagram[6] for the operator with largest null space.

(3) The path sum formula agress with all previously determined special cases of the stretched denominator function. In particular, it agrees with all U(2) denominator functions, with all stretched U(3) denominator functions,[7] and with all stretched U(n) denominator functions for the adjoint-type operators.[8]

(4) Stretched patterns in U(3) have the difference $\Gamma_{12}-\Gamma_{22}$ as large as possible, consistent with $\Delta(\Gamma)$ fixed. "Stretching" in U(n) is less geometrically obvious and corresponds to having the largest dimension of the irrep denoted by $[\Gamma_{1k} \ldots \Gamma_{kk}]$ for each k = 1, 2, . . . n-1, consistent with the constraints of lexicality and $\Delta(\Gamma)$ fixed. It can be shown that this operator pattern--denotes by (Γ_s)--is unique.

(5) The denominator function determined by the path sum formula has the general form:

$$[\text{denom. fcn.}]^{-2} = \frac{\text{polynomial}}{\text{product of linear factors}}$$

The polynomials determined in this way constitute a new family of special functions with remarkable properties, currently under investigation.[9] These polynomials are invariant under the original U(n) group (as is obvious) and are totally symmetric under the S(n) group permuting the variables $\{p_{in}\}$. In barycentric n-space, the polynomials are positive in the lexical region and characterized (in the lexical region) by a simplex of zeroes.

(6) It is remarkable that the operator $\left\langle \begin{smallmatrix} (\Gamma_s) \\ [M] \end{smallmatrix} \right\rangle$ itself is determined by knowledge of the denominator function. This corresponds to the fact that boundary Racah functions (those relating to maximal final irreps) are monomials determined by the corresponding (known) denominator functions. Hence Racah -coupling of the sequence of elementary operators in the Cartan product suffices.

(7) As a concluding remark, let us observe that knowledge (in principle) of the maximal null space operator allows us to remove this operator from the sum over Γ patterns in equation (18). In principle, therefore we are in a position to <u>iterate</u> the process that led to determining the (Γ_s) operator, thereby determining the next operator in the sequence, etc. This iteration process--though basically correct and feasible--is not yet fully understood and no claim can be made, as yet, to having completely resolved the general multiplicity problem.

FOOTNOTES

1. Canonical here means no free choice to within equivalence under the Weyl group.

2. The characteristic null space consists of complete irreps; this distinction eliminates from consideration accidental vanishings on particular vectors (which are basis dependent) and known to occur.

3. An extremal Gel'fand pattern is one in which the weights are a permutation of the irrep labels [m].

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